Geometry and Topology of SO(4) Trivializable Gauge Fields

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The geometric and topological properties of SO(4)-trivializable SU(2) gauge fields are investigated in detail. The geometry of trivializable configurations is most adequately described by a conformally flat Riemannian structure, which yields a common geometric framework for both instantons and merons. In topologically nontrivial situations, the trivializable gauge fields exhibit a number of point defects (merons), which may be characterized by a quantized defect charge. The SO(2) reduction of the corresponding SU(2) bundle yields monopole configurations of the t'Hooft-Polyakov type.

I. Introduction

There are numerous investigations of classical solutions to non-abelian field equations in the recent literature. This evidently demonstrates the great interest for a better understanding of the geometric and topological peculiarities inherent in the classical configurations. In view of the importance which the classical solutions bear also for the quantized version of the corresponding field theory (instanton effects etc.), such a great interest seems extremely plausible.

The present paper also deals with a classical effect; this is the SO(4) trivializability of SU(2) gauge fields. The class of SO(4) trivializable gauge fields embraces such important examples as merons and monopoles; however, besides the search for special examples it seems also desirable to investigate the trivializability phenomenon from a more general and abstract point of view. Indeed, the trivializable gauge fields exhibit several peculiar geometric and topological features which shall be the subject of the subsequent study. We thereby continue the program as represented in the preceding paper [1] to which the reader is referred for notations and coventions being not repeated here. The paper is organized as follows:

In Sect. II, the conformal geometry of trivializable gauge fields is represented in full detail. The motivation for describing the trivializable configurations in terms of a conformally flat Riemannian structure originates from the use of a certain gauge ("characteristic gauge"). In this gauge, the curvature $\Phi[\alpha]$ of the trivializable connection α is cast into a SU(2) reduction of some SO(4) tensor b, the Weylian $W[b]$ of which vanishes identically. Since the Weyl tensor of any conformally flat Riemannian also vanishes, it is strongly suggestive to express the geometry of trivializable gauge fields in terms of such a conformally flat Riemannian structure. The implications of the conformal ansatz are discussed in great detail as well as the geometric relationship to the conformal properties of self-dual fields (instantons).

In Sect. III, the topological point defects of trivializable connections are discussed with respect to the problem how to define the defect charge uniquely. We thereby introduce the notion of a "Gauß current", which is an extrinsic, gauge invariant counterpart of the intrinsic "Chern current". Since the latter one is gauge dependent, it cannot be used to characterize uniquely the topological point defects. On the other hand the flux lines of the Gauß current agree with the characteristic lines and the defect charge is the total flux of the Gauß current through any closed 3-surface around the defect position. The pairing of defects is shown to occur only in highly symmetric arrangements.

In Sect. IV, the "generating section" of the trivializable connection is used to reduce the structure group SO(3) to the abelian subgroup SO(2). It is shown that the generating section must neces-

sarily be singular on a stringlike submanifold of the base space. Thereby, a topological string flux emerges which is due to a pointlike abelian charge (monopole), where the string may be interpreted as the world line of the monopole. The defect charges of the embedding SO(3) bundle arise, in this picture, as the discontinuities of the string flux.

In Sect. V, a special example of the reduction SO(3) → SO(2) of the preceding section is given. The monopole field strength is the Euler class of the reduced SO(2) bundle with the quantized monopole charge coinciding with the Euler number. Concretely, the t'Hooft-Polyakov solution to the coupled Yang-Mills-Higgs system is studied and found to be the only trivializable configuration allowed by that field equations. From the topological point of view, the corresponding gauge field is due to a "half-meron".

II. Conformal Properties of Trivializable Gauge Fields

According to the general definition [1], a trivializable SO(3) principal bundle ($\bar{\Sigma}_4$, say) can always be embedded into a trivial SO(4) bundle ($\mathring{\Lambda}_4$). In the following, we study in detail the most important properties following from the embedding $\bar{\Sigma}_4 \to \mathring{\Lambda}_4$. Especially, we want to elaborate the conformal geometry of trivializable gauge fields. A comparison of the conformal properties of the (trivializable) merons and the (non-trivializable) instantons yields some insight into the common and the different geometric features of both types of solutions.

But first let us adapt the trivializability conditions to the present purpose.

II.1 Trivializability Conditions

Since any trivial bundle ($\mathring{\Lambda}_4$) is a product bundle over the base space E_4^-,

$$\mathring{\Lambda}_4 = E_4^- \times SO(4),\tag{II.1}$$

one can always introduce the canonical connection ($\mathring{\omega}$, say) in that product bundle and hence the local connection 1-form $\bar{\alpha}$ for the reduced bundle is just the $\mathfrak{SO}(3)$ restriction of the canonical connection $\mathring{\omega}$ in ($\mathring{\Lambda}_4$):

$$\bar{\alpha} = \mathring{\omega}\,|_{\mathfrak{SO}(3)}.\tag{II.2}$$

This reduction mechanism enables one to represent any trivializable α as some 3-distribution $\bar{\Delta}$ in Euclidean four-space E_4^-, as was demonstrated for the special case of the di-meron configuration.

The most striking feature of any trivializable connection is the fact that there exists a tensor object β transforming homogeneously with respect to SU(2) gauge transformations

$$\beta' = X^{-1}\beta X,\tag{II.3}$$

whereas α obeys the usual inhomogeneous law

$$\alpha' = X^{-1}\alpha X + X^{-1}\,\mathbf{d}X.\tag{II.4}$$

Concerning the notations, let us remark that once the trivializable SO(3) bundle has been generated by the reduction mechanism (II.2), we may evoke the isomorphism $\mathfrak{SO}(3) \sim \mathfrak{SU}(2)$ and will henceforth consider SU(2) as the proper gauge group. In this sense, we think of the $\mathfrak{SU}^+(2)$ valued 1-form β to be decomposed as

$$\beta = \mathbb{B}_i\,X^i = B_{i\mu}\,X^i\,\mathbf{d}x^\mu.\tag{II.5}$$

The gauge transformation (II.3) then reads in components

$$\mathbb{B}'_i = \mathbb{B}_j\,S^j{}_i,\tag{II.6}$$

where the SO(3) matrix S is the adjoint representation of the $SU^+(2)$ element X. A similar decomposition holds for the $\mathfrak{SU}^+(2)$ connection α (in place of the $\mathfrak{SO}(3)$ connection $\bar{\alpha}$).

In geometric terms, the tensor object β plays the role of extrinsic curvature for the 3-distribution $\bar{\Delta}$ mentioned above. Further, the β fields also enter the curvature $\Phi[\alpha]$ of the trivializable α, which emerges as a 2-form built by the wedge product of the extrinsic curvature 1-form [2]

$$\Phi[\alpha] = -\beta \wedge \beta,\tag{II.7}$$

$$(\Phi[\alpha] := \mathbf{d}\alpha + \alpha \wedge \alpha).\tag{II.7a}$$

In this way, the Bianchi identity becomes immediately obvious:

$$D\Phi \equiv \mathbf{d}\Phi + \alpha \wedge \Phi - \Phi \wedge \alpha = 0\tag{II.8}$$

because the extrinsic curvature β is covariantly constant:

$$D\beta = \mathbf{d}\beta + \alpha \wedge \beta + \beta \wedge \alpha = 0.\tag{II.9}$$

The (necessary and sufficient) conditions (II.7 + 9) for a given $\mathfrak{SU}^+(2)$ connection α to be trivializable

may be rephrased also in the following way: for any trivializable α there exists a covariantly constant object β such that the modified connection $\mathring{\alpha}$

$$\mathring{\alpha} = \alpha \pm \beta \qquad (\text{II.10})$$

is trivial (i.e. its curvature $\mathring{\Phi}$ (II.7a) vanishes).

In the following, we are using four different gauges:

(i) the positive gauge for which the connection α^+ agrees with its associated tensor object β^+:

$$\alpha^+ = \beta^+, \qquad (\text{II.11})$$

(ii) the negative gauge which is characterized by

$$\alpha^- = -\beta^- \quad \text{and} \qquad (\text{II.12})$$

(iii) the neutral gauge α^n, which is a sort of (matrix) mean value of the positive and negative gauges.

(iv) The fourth gauge ("characteristic gauge") is that gauge for which the coefficients $\mathbb{B}_i$ of the exterior curvature β are expressed in terms of the characteristic vector field p as follows:

$$^{\times}B_{i\mu} = 2 c^{-1} (X_i)^\lambda{}_\mu p_\lambda. \qquad (\text{II.13})$$

We do not know wether any trivializable α admits such a gauge; however, we want to use (II.13) hereafter for the construction of the corresponding connection coefficients $^{\times}A_{i\mu}$ in order to demonstrate how the trivializable gauge fields favour a conformally flat Riemannian geometry.

II.2 The Conformal Geometry

In order to get some restrictive conditions upon the possible shape of the trivializable $^{\times}\alpha$ ($= {}^{\times}\mathbb{A}_i X^i$) we first look at the trivializability condition (II.7), which becomes by use of (II.13)

$$^{\times}\mathbb{F}_i = -\tfrac{1}{2} \varepsilon_i{}^{jk} \, {}^{\times}\mathbb{B}_j \wedge {}^{\times}\mathbb{B}_k = -(X_i)^{\beta\alpha} \mathbf{b}_{\alpha\beta} \qquad (\text{II.14})$$

$$(\Phi = \mathbb{F}_i X^i).$$

Here, the $\mathfrak{SO}(4)$-valued 2-form $\mathbf{b} = \{\mathbf{b}^\alpha{}_\beta\}$ is given in terms of the characteristic vector p as follows:

$$\mathbf{b}_{\alpha\beta} = \tfrac{1}{2} b_{\alpha\beta\mu\nu} \, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu, \qquad (\text{II.15a})$$

$$b_{\alpha\beta\mu\nu} = (c\,p)^{-2} [\hat{P}_{\alpha\mu} \, \hat{P}_{\beta\nu} - \hat{P}_{\alpha\nu} \, \hat{P}_{\beta\mu}],$$

$$(p^2 := -p^\lambda p_\lambda). \qquad (\text{II.15b})$$

The (Euclidean) projector $\hat{P}$ annihilates the characteristic vector p:

$$\hat{P}_{\mu\nu} = g_{\mu\nu} + \hat{p}_\mu \hat{p}_\nu \quad (\hat{p}_\mu := p^{-1} p_\mu). \qquad (\text{II.16})$$

Since the $\mathfrak{SO}(4)$ element $\mathbf{b}$ is a skew-symmetric construction of the symmetric object $\hat{P}$, it leads to a vanishing Weyl tensor $W[b]$

$$W_{\alpha\beta\mu\nu} = b_{\alpha\beta\mu\nu} + \tfrac{1}{6} b \, [g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}]$$

$$- \tfrac{1}{2} [g_{\alpha\mu} b_{\beta\nu} - g_{\beta\mu} b_{\alpha\nu} - g_{\alpha\nu} b_{\beta\mu} + g_{\beta\nu} b_{\alpha\mu}]$$

$$= 0, \quad (b_{\beta\nu} := b^\mu{}_{\beta\mu\nu} = g^{\alpha\mu} b_{\alpha\beta\mu\nu}). \qquad (\text{II.17})$$

But, because a vanishing Weyl tensor is characteristic for a conformally flat Riemannian geometry, it is strongly suggestiv to introduce the conformal metric

$$G_{\mu\nu} = \chi^2 g_{\mu\nu}. \qquad (\text{II.18})$$

Further, one tries to obtain the connection coefficients $^{\times}\mathbb{A}_i$ as the $\mathfrak{SU}^+(2)$ restriction of the Levi-Civita connection Γ due to this conformally flat metric:

$$^{\times}A_{i\mu} = (X_i)^\beta{}_\alpha \, \Gamma^\alpha{}_{\beta\mu}, \qquad (\text{II.19a})$$

$$\Gamma^\alpha{}_{\beta\mu} = \tfrac{1}{2} G^{\alpha\lambda} (\partial_\beta G_{\lambda\mu} + \partial_\mu G_{\lambda\beta} - \partial_\lambda G_{\beta\mu}) \qquad (\text{II.19b})$$

$$= g^\alpha{}_\beta q_\mu + g^\alpha{}_\mu q_\beta - g_{\beta\mu} q^\alpha \qquad (\text{II.19c})$$

$$(q_\lambda := \partial_\lambda \ln \chi).$$

With this ansatz for $^{\times}\alpha$, the corresponding $\mathfrak{SU}^+(2)$ curvature $^{\times}\Phi = \Phi[^{\times}\alpha]$ becomes just the $\mathfrak{SU}^+(2)$ restriction of the $\mathfrak{SO}(4)$ curvature $\mathbb{R}[\Gamma]$, in components [3]

$$^{\times}\mathbb{F}_i = (X_i)^\beta{}_\alpha \, \mathbb{R}^\alpha{}_\beta, \qquad (\text{II.20a})$$

$$\mathbb{R}^\alpha{}_\beta = \mathbf{d}\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\gamma \wedge \Gamma^\gamma{}_\beta, \qquad (\text{II.20b})$$

$$(\mathbb{R}^\alpha{}_\beta \equiv \tfrac{1}{2} R^\alpha{}_{\beta\mu\nu} \, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu).$$

The reduction formula (II.20a) says that the curvature Φ of the $\mathfrak{SU}^+(2)$ projection α of Γ is just the projection of the $\mathfrak{SO}(4)$ curvature $\mathbb{R}$ of Γ. In general, both processes of forming the projection and curvature will not commute; otherwise one could not obtain trivializable gauge fields. But in the present case the symmetric part γ of Γ,

$$\gamma = \tfrac{1}{2} (\Gamma + \Gamma^{\mathrm{T}}) \quad (\Gamma^{\mathrm{T}} \text{ is the transpose of } \Gamma) \qquad (\text{II.21})$$

is proportional to the identity

$$\gamma^\alpha{}_{\beta\mu} = g^\alpha{}_\beta q_\mu, \qquad (\text{II.22})$$

and therefore the generally valid reduction formula

$$\Phi = (\mathbb{R} - \gamma \wedge \gamma)\,|_{\mathfrak{SU}^+(2)} \qquad (II.23)$$

reduces to the special result (II.20b).

By virtue of these reductive properties of the $\mathfrak{Gl}(4, \mathbb{R})$ connection Γ, we are able to reformulate the problem completely in terms of the Riemannian conformal structure. First, the gauge covariant derivative of the tensor object β

$$D_\mu B_{iv} = \partial_\mu B_{iv} + \varepsilon_i{}^{jk} A_{j\mu} B_{kv} \qquad (II.24)$$

may be extended to a generally (i.e. gauge and coordinate) covariant derivative $\mathscr{D}$:

$$\mathscr{D}_\mu B_{iv} := D_\mu B_{iv} - \Gamma^\lambda{}_{v\mu} B_{i\lambda}. \qquad (II.25)$$

Next, the generally covariant constancy of the $\mathfrak{SU}^+(2)$ generators X^i,

$$\mathscr{D}_\lambda (X_i)^\mu{}_v = 0\,, \qquad (II.26)$$

may be used to transform the derivative of the extrinsic curvature $^\times B_{i\mu}$ (II.25) into the coordinate covariant derivative of the characteristic vector p (cf. II.13):

$$\mathscr{D}_\mu B_{iv} = 2\,c^{-1}(X_i)^\lambda{}_v(\nabla_\mu p_\lambda)\,, \qquad (II.27\,a)$$

$$(\nabla_\mu p_\lambda := \partial_\mu p_\lambda - \Gamma^\sigma{}_{\lambda\mu} p_\sigma)\,. \qquad (II.27\,b)$$

Hence, the gauge covariant trivializable condition (II.9) upon the extrinsic curvature β is converted easily into the corresponding coordinate covariant condition upon the characteristic vector p [4]:

$$\nabla_\mu p_v = 0\,. \qquad (II.28)$$

Next, we want to express also the first condition (II.7) in terms of the Riemannian geometry which then simply becomes by combining (II.14) with (II.20)

$$\mathbb{R}^\alpha{}_\beta = -\,\mathbf{b}^\alpha{}_\beta\,. \qquad (II.29)$$

The Riemannian $\mathbb{R}$ on the left-hand side may be expressed further in terms of the conformal gradient q (II.19c), whereas the $\mathfrak{SO}(4)$ tensor $\mathbf{b}$ on the right-hand side has to be substituted in its coordinate covariant version [3]

$$b^\alpha{}_{\beta\mu\nu} = -\,(c\,\dot{p})^{-2}[\hat{P}^\alpha{}_\mu \hat{P}_{\beta\nu} - \hat{P}^\alpha{}_\nu \hat{P}_{\beta\mu}]\,, \qquad (II.30)$$

$$(\dot{p}^2 := G^{\mu\nu} p_\mu p_\nu)\,.$$

Thus, the trivializability condition (II.29) yields the following coupling of both vector fields q, p:

$$\partial_\nu q_\mu = q_\nu q_\mu + c^{-2} p_\nu p_\mu - \tfrac{1}{2} g_{\nu\mu}(q^\lambda q_\lambda + c^{-2} p^\lambda p_\lambda)\,. \qquad (II.31)$$

On the other hand, the covariant constancy of p (II.28) reads in non-covariant form

$$\partial_\nu p_\mu = p_\nu q_\mu + q_\nu p_\mu - g_{\nu\mu}(p^\lambda p_\lambda)\,. \qquad (II.32)$$

The two equations (II.31, 32) may be decoupled by means of the substitution

$$u^\nu = \tfrac{1}{2}(q^\nu + c^{-1} p^\nu)\,, \qquad (II.33\,a)$$

$$v^\nu = \tfrac{1}{2}(q^\nu - c^{-1} p^\nu)\,, \qquad (II.33\,b)$$

and then both vector fields u, v satisfy the same equation, namely $(w \to u, v)$

$$\partial_\nu w_\mu = 2 w_\nu w_\mu - g_{\nu\mu}(w^\lambda w_\lambda)\,. \qquad (II.34)$$

The non-trivial solutions to the problem (II.34) constitute an important example of a trivializable $\mathfrak{SU}(2)$ connection. It is the well-known di-meron solution to the free Yang-Mills equations, which has been studied in great detail and therefore needs not be repeated here [1]. But there is a point which deserves some futher attention: the covariant constancy of the characteristic vector p (II.28) implies the constancy of the extrinsic curvature according to (II.27)

$$\mathscr{D}_\mu B_{iv} = 0\,. \qquad (II.35)$$

On the other hand, the curvature $\Phi[\alpha]$ is composed quadratically of the B fields (cf. II.14). Therefore, the curvature coefficients $\mathbb{F}_i$ are also covariantly constant (in the general sense):

$$\mathscr{D}_\lambda F_{i\mu\nu} = 0\,. \qquad (II.36)$$

This is a far more restrictive condition than both the Bianchi identity (II.8) and the Yang-Mills equations

$$\mathscr{D}_\mu F_i{}^{\mu\nu} = 0 \qquad (II.37)$$

are imposing upon the curvature Φ. Clearly, the latter equations follow from the more restrictive ones (II.36).

The geometric meaning of (II.36) becomes clear when one rewrites this condition within the framework of the embedding Gl(4, $\mathbb{R}$) bundle. Because of the identity

$$\mathscr{D}_\lambda F_{i\mu\nu} \equiv (X_i)^\beta{}_\alpha \nabla_\lambda R^\alpha{}_{\beta\mu\nu} \qquad (II.38)$$

the constancy of the $\mathfrak{SU}^+(2)$ curvature $\mathbb{F}_i$ (II.36) implies the constancy of the $\mathfrak{SO}(4)$ curvature $\mathbb{R}$:

$$\nabla_\lambda R_{\alpha\beta\mu\nu} = 0, \qquad (\text{II.39})$$

which means that the Riemannian space involved here is a locally symmetric one. So we see that the emergence of a symmetric space is responsible for the reduction of the order of the differential equations: in place of solving the second order Yang-Mills equations (II.37) for the gauge potentials $A_{i\mu}$, one had merely to solve the coupled first order problem of the trivializability conditions (II.7 + 9) which reduced to the single first order equation (II.34) in case of our conformal ansatz; the field equations (II.37) were then satisfied automatically because the first order problem led to a locally symmetric Riemannian structure.

This reduction mechanism with respect to the order of the differential equations involved resembles somewhat the analogous effect of the self-duality conditions leading to the well-known instanton solutions. Indeed, there are even more far reaching relationships between the meron and instanton types of solutions, if one considers the problem from the geometric point of view. In this approach, the unifying structure of instantons and merons turns out as a symmetric Riemannian space of the conformal type.

II.3 The General Conformal Ansatz [5–9]

Since the well-known instanton solutions to the free Yang-Mills equations fit also into the conformal ansatz (II.19 a), one is led to the supposition that this ansatz embraces a wider class of solutions than are represented by the meron and instanton types alone. The striking feature of the conformal ansatz is that it reduces the order of the differential equations by leading to a first integral in a very natural way. Hence it is closely related to both the self-duality condition and the condition of a symmetric space. Indeed, as we shall see readily, the single instanton solution is self-dual and simultaneously is due to a symmetric Riemannian space.

First, consider the transcription of the (generally covariant) Yang-Mills equations (II.37) into the Riemannian SO(4) structure which yields

$$\nabla_\mu R_{\alpha\beta}{}^{\mu\nu} = 0. \qquad (\text{II.40})$$

The Bianchi identity for the Riemannian $\mathbb{R}$ reads

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0. \qquad (\text{II.41})$$

Therefore, contracting this identity by $G^{\alpha\lambda}$ yields on account of the field equations (II.40) the following condition upon the Ricci tensor $R_{\lambda\mu}$:

$$\nabla_\nu R_{\lambda\mu} - \nabla_\mu R_{\lambda\nu} = 0. \qquad (\text{II.42})$$

A further contraction shows that the divergence of the Ricci tensor equals the gradient of the curvature scalar $R\,(= G^{\mu\nu} R_{\mu\nu})$:

$$\partial_\nu R = \nabla_\mu R^\mu{}_\nu. \qquad (\text{II.43})$$

On the other hand, the contraction of the field equations (II.40) gives vanishing divergence of the Ricci tensor. Therefore, the curvature scalar must be constant. If this constancy condition upon the curvature scalar R is expressed in terms of the conformal factor χ (cf. (II.18)), one is led to the scalar equation

$$\partial^\mu \partial_\mu \chi + \tfrac{1}{6} R \chi^3 = 0 \quad (R = \text{const}). \qquad (\text{II.44})$$

This is a sort of first integral for the original Yang-Mills problem because it is a second order equation for χ despite the fact that χ enters the connection coefficients $^\times A_{i\mu}$ (II.19) via its first derivative!

Clearly, any solution of the conformal type must satisfy the scalar equation (II.44). Especially, those conformal solutions which are due to a symmetric space have a priori a constant curvature scalar R and therefore form a special subset of solutions to the general conformal equation (II.44). For the preceding example of the di-meron configuration, the conformal factor has been identified as the (Euclidean) length of the characteristic vector $\boldsymbol{p}$:

$$\chi = |\boldsymbol{p}| = \frac{|\boldsymbol{a} - \boldsymbol{b}|^2}{|\boldsymbol{x} - \boldsymbol{a}| \cdot |\boldsymbol{x} - \boldsymbol{b}|}. \qquad (\text{II.45})$$

This is a solution of (II.44) with the meron positions being located at $\boldsymbol{a}$ and $\boldsymbol{b}$, resp., and the curvature scalar assuming the value

$$R = -6 |\boldsymbol{a} - \boldsymbol{b}|^{-2}. \qquad (\text{II.46})$$

We do not know the most general solution to (II.44) which is due to a symmetric space but we can show that at least the single instanton also belongs to this class (besides the di-meron configuration).

II.4 Relationship to Instantons

The way in which the (anti) self-duality conditions select a special subset of solutions to the general conformal equation (II.44) is most conveniently described in terms of the Riemannian structure.

According to the symmetry of the Riemannian $\mathbb{R}$:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu}, \qquad (II.47)$$

it may be decomposed into a sum of four terms each of which has a definite duality property:

$$\mathbb{R} = {}^+\mathbb{R}^+ + {}^+\mathbb{R}^- + {}^-\mathbb{R}^+ + {}^-\mathbb{R}^-. \qquad (II.48)$$

The individual terms on the right are defined by means of the (anti) self-duality projectors Π as

$$
{}^+R^+_{\alpha\beta\mu\nu} = \overset{+}{\Pi}{}_{\alpha\beta}{}^{\lambda\sigma} R_{\lambda\sigma\varrho\varkappa} \overset{+}{\Pi}{}^{\varrho\varkappa}{}_{\mu\nu},
$$
$$
{}^+R^-_{\alpha\beta\mu\nu} = \overset{+}{\Pi}{}_{\alpha\beta}{}^{\lambda\sigma} R_{\lambda\sigma\varrho\varkappa} \overset{-}{\Pi}{}^{\varrho\varkappa}{}_{\mu\nu}, \quad \text{etc.} \qquad (II.49)
$$

The (anti) self-duality projectors $(\overset{-}{\Pi})\ \overset{+}{\Pi}$ themselves may be expressed by the (anti) self-dual $\mathfrak{SU}^\pm(2)$ generators $(Y^i)\ X^i$ in the following way:

$$\overset{+}{\Pi}_{\mu\nu\varrho\sigma} = -(X^i)_{\mu\nu}(X_i)_{\varrho\sigma}, \qquad (II.50a)$$

$$\overset{-}{\Pi}_{\mu\nu\varrho\sigma} = -(Y^i)_{\mu\nu}(Y_i)_{\varrho\sigma}. \qquad (II.50b)$$

Since the (anti) self-dual projectors are orthogonal,

$$\overset{+}{\Pi}_{\mu\nu}{}^{\varrho\sigma}\overset{-}{\Pi}_{\varrho\sigma\lambda\varkappa} = 0, \qquad (II.51)$$

the reduction formula (II.19) restricts the Riemannian connection Γ to its (self-dual) $\mathfrak{SU}^+(2)$ part and a similar observation holds for the restriction of $\mathbb{R}[\Gamma]$ to its $\mathfrak{SU}^+(2)$ subcurvature Φ (II.20). Therefore, if we want to have a (anti) self-dual $\mathfrak{SU}^\pm(2)$ curvature Φ ($*\Phi = \pm \Phi$), we have to require

$$\quad {}^+\mathbb{R}^- = 0 \quad \text{(self-dual)}, \qquad (II.52a)$$

$$\quad {}^+\mathbb{R}^+ = 0 \quad \text{(anti self-dual)}. \qquad (II.52b)$$

If the conformally flat Riemannian $\mathbb{R}[\Gamma]$ (II.20b) is introduced here, one finds the two conditions

$$R_{\mu\nu} = \tfrac{1}{4} R G_{\mu\nu} \quad \text{(self-dual)}, \qquad (II.53a)$$

$$R \ = 0 \qquad \text{(anti self-dual)}. \qquad (II.53b)$$

The latter case (b) cuts down the general conformal equation (II.44) to the linear Laplace equation

$$\partial^\mu \partial_\mu \chi = 0, \qquad (II.54)$$

which leads to the t'Hooft description of multi-instanton solutions [5]. In the present context, the self-dual case is more interesting because (II.53a) says that the underlying geometry is due to conformally flat Einstein space which automatically is a symmetric space [10].

The solution to the self-duality problem (II.53a) is readily obtained by expressing that equation in terms of the conformal gradient q (II.19), which yields an equation quite similar to the "di-meron equation" (II.34), namely

$$\partial_\mu q_\nu = q_\mu q_\nu + \tfrac{1}{4} g_{\mu\nu}(\partial_\lambda q^\lambda - q^\lambda q_\lambda). \qquad (II.55)$$

One reads off from this equation that the integral curves of q must be straight lines, so that the only non-trivial solution is the hedge hog field

$$q = \frac{2x}{\lambda^2 + |x|^2}, \qquad (II.56a)$$

where λ is an integration constant and related to the curvature scalar R through

$$R = -\frac{48}{\lambda^2}. \qquad (II.56b)$$

As a consequence, the gauge potential ${}^\times A_{i\mu}$ (II.19) acquires the well-known form for the single instanton solution

$$^\times\alpha = \frac{|x|^2}{\lambda^2 + |x|^2}\, X^{-1}\, \mathbf{d}X \quad (X = -\hat{x}_\mu \overset{+}{\Xi}{}^\mu). \qquad (II.57)$$

So we see that there is an intimate geometric relationship between the di-meron and the single instanton solutions, if they are considered as embeddings into a Riemannian SO(4) structure. However, the symmetric space structure of the instanton is immediately evident because it is the stereographic projection [11] of a 4-dimensional sphere S^4 (which is clearly a symmetric space) onto the Euclidean space E$_4$. In contrast to this fact, the di-meron case seems to escape such a simple interpretation.

III. Topological Point Defects

The topological properties of an arbitrary SU$^+$(2) bundle connection α are usually described in terms

of its second Chern class [12] $*\varrho_c[\alpha]$,

$$*\varrho_c[\alpha] = -\frac{1}{16\pi^2} \operatorname{tr}\{\Phi \wedge \Phi\}, \qquad (\text{III.1})$$

where Φ is the curvature (II.7a) of α. From the topological point of view, the most striking feature of a trivializable α is the fact that its Chern class vanishes identically:

$$*\varrho_c[\alpha] = -\frac{1}{16\pi^2} \operatorname{tr}\{\beta \wedge \beta \wedge \beta \wedge \beta\} \equiv 0. \qquad (\text{III.2a})$$

Clearly, one must be somewhat careful what happens at the singularities of β [13]. For the present purpose, we shall assume that β is singular on a set of isolated points $\mathcal{M} = \bigcup_a M_a$ of E_4. In this way the base space of the bundles involved here becomes $E_4^- = E_4 \setminus \mathcal{M}$ (cf. II.1), and the "topological charge" density $\varrho_c(x)$ will be localized on $\mathcal{M}$.

As a consequence of the vanishing of the Chern class $*\varrho_c$ (III.2a), the trivializable connections α cannot be classified as usual by the (vanishing) Chern number Q if the latter one is understood, as usual, as a global topological property of α:

$$Q = \int_{E_4^-} *\varrho_c \to 0. \qquad (\text{III.2b})$$

On the other hand, it is just the vanishing of $*\varrho_c$ which offers a possibility to characterize the singularities of α. For, if we apply Stokes theorem to the integral (III.2b), running over a piece of E_4 with boundary ∂E_4, we see that the value of the boundary integrals do not depend upon the special shape of that boundary. Therefore, the boundary integrals lend themselves to a local topological characterization of the "defects" of α residing upon the singular set $\mathcal{M}$. For a concretization of this idea we have to look for a suitable 3-current $*j$, which has the Chern class $*\varrho_c$ as its source. First, we demonstrate that the intrinsic bundle geometry cannot provide us with an acceptable candidate (cf. Chern current). Rather, the solution to the present problem will be obtained by resorting to the extrinsic geometry (cf. Gauß current).

III.1 Chern Current

By virtue of Weyl's homomorphism [12], the general Chern class $*\varrho_c$ (III.1) is an element of the fourth cohomology group $H^4(E_4^-, \mathbb{Z})$ and hence is

closed over E_4^- (independently of the validity of (III.2a)):

$$\mathbf{d} *\varrho_c = 0. \qquad (\text{III.3})$$

Therefore, $*\varrho_c$ may be represented locally as the exterior differential of a certain 3-form $*j_c$ ("Chern current"):

$$*\varrho_c[\alpha] = \mathbf{d} *j_c[\alpha], \qquad (\text{III.4})$$

where

$$*j_c[\alpha] = -\frac{1}{16}\pi^{-2} \operatorname{tr}\{\alpha \wedge \mathbf{d}\alpha + \tfrac{2}{3}\alpha \wedge \alpha \wedge \alpha\}$$

$$= -\frac{1}{16\pi^2} \operatorname{tr}\{\alpha \wedge \Phi - \tfrac{1}{3}\alpha \wedge \alpha \wedge \alpha\}. \qquad (\text{III.5})$$

Contrary to the Chern class $*\varrho_c$, the corresponding current $*j_c$ is not gauge invariant but transforms according to

$$*j_c' = *j_c + \frac{1}{16\pi^2} \mathbf{d} \operatorname{tr}\{\alpha \wedge \zeta^-\} + \frac{1}{2\pi^2} \mathbf{d}S^3 \qquad (\text{III.6})$$

with respect to the gauge transformation (II.4). Here, we have introduced the pure gauge terms $\zeta^\pm$ through

$$\zeta^+ = X^{-1} \cdot \mathbf{d}X = \zeta_i^+ X^i, \qquad (\text{III.7a})$$

$$\zeta^- = X \cdot \mathbf{d}X^{-1} = \zeta_i^- X^i. \qquad (\text{III.7b})$$

Their coefficients $\zeta_i^\pm$ are related to each other by the adjoint representation $S[X]$ of the $SU^+(2)$ element X:

$$\zeta_i^+ = -S^j{}_i \zeta_j^-, \qquad (\text{III.8})$$

$$(S^i{}_j[X] = \operatorname{tr}\{X^{-1} X^i X X_j\}).$$

Parametrizing the gauge element X by the unit vector field $\hat{k}$ ($\hat{k}^\mu \hat{k}_\mu = -1$),

$$X = -\hat{k}_\mu \overset{+}{\varXi}{}^\mu, \qquad (\text{III.9})$$

the coefficients $\zeta_i^\pm$ are found as

$$\zeta_i^+ = 4(X_i)_{\alpha\beta} \hat{k}^\alpha \mathbf{d}\hat{k}^\beta, \qquad (\text{III.10a})$$

$$\zeta_i^- = 4(Y_i)_{\alpha\beta} \hat{k}^\alpha \mathbf{d}\hat{k}^\beta. \qquad (\text{III.10b})$$

Obviously, the gauge matrix X (III.9) induces a map $[\hat{k}]$: $E_4^- \to SU^+(2)$; and the last term on the right of (III.6) is just the pullback to E_4^- of the

invariant volume element of $SU^+(2)$ with respect to $[\hat{k}]$

$$\mathbf{d}S^3 = \frac{1}{4!}\operatorname{tr}\{\zeta^+ \wedge \zeta^+ \wedge \zeta^+\}. \quad \text{(III.11)}$$

Since the gauge group $SU^+(2)$ is topologically equivalent to the unit sphere S^3, the pullback $\mathbf{d}S^3$ (III.11) represents the 3-cell on the unit sphere S^3:

$$\mathbf{d}S^3[\hat{k}] = \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta}\,\hat{k}^\alpha(\mathbf{d}\hat{k}^\beta) \wedge (\mathbf{d}\hat{k}^\gamma) \wedge (\mathbf{d}\hat{k}^\delta). \quad \text{(III.12)}$$

The peculiar transformation behaviour of the Chern current, if applied to trivializable gauge fields, immediately implies some inconsistency, through which one is led to refuse that current as a possible candidate for characterizing the topological defects of α.

III.2 Ambiguity of the Topological Charge

The Chern current $*j_c[\alpha]$ (III.5) is normalized in such a way that for a trivial connection $\mathring{\alpha}$ (e.g. $\mathring{\alpha} = \zeta^+$, $\Phi[\mathring{\alpha}] = 0$) it just becomes the 3-cell on S^3 (apart from the surface area of S^3, $2\pi^2$, as a normalizing factor):

$$*j_c[\zeta^+] = \frac{\mathbf{d}S^3[\hat{k}]}{2\pi^2}. \quad \text{(III.13)}$$

On the other hand, we could have first started from a vanishing connection $\alpha = 0$ and would then have performed the gauge transformation leading to $\alpha = \zeta^+$ according to the transformation law (II.4). On account of the variation of $*j_c$ during a gauge transformation, the transformed current $*j'_c$ becomes identical to the result (III.13). This fact has important consequences when we consider the value $Q[C^3]$ of the current $*j_c$ upon some 3-cycle $C^3 \subset E_4^-$

$$Q[C^3] = \oint_{C^3} *j_c. \quad \text{(III.14)}$$

In the first case $(\alpha = 0)$, we clearly would have obtained $Q[C^3] = 0$. In the second case however $(\alpha = \zeta^+)$, we may obtain an arbitrary integer for Q if the restriction of the map $[\hat{k}]$ to C^3 is continuous over C^3. We do not demand that the map $[\hat{k}]$ be continuous over the whole E_4; rather, we admit a singularity of the unit vector field $\hat{k}(x)$ inside that piece of E_4 which is bounded by C^3. In this way, one can arbitrarily change the "topological charge" Q (III.14), enclosed by C^3, by an appropriate gauge

transformation, which induces some element of the third homotopy group $\Pi_3(C^3) = \mathbb{Z}$ upon any C^3 not containing the singularities of the vector field $\hat{k}$.

Applying this result to the present class of trivializable connections, it is obvious that the Chern current cannot be used to characterize the topological defect of α via (III.14). For one can always change the topological charge Q of such a defect by letting coincide the singularities of the vector field $\hat{k}(x)$ with the singular points $M_a \in \mathscr{M}$ of the trivializable α. In this case, the gauge transformations may be smooth over the whole base space E_4^- but nevertheless Q is not a topological invariant.

Hence, the object Q, as it is defined through (III.14), is not a meaningful characterization of the topological defects, though it has been used frequently in the literature [14]. A simple example shall demonstrate this failure of the Chern current.

III.3 Example: Di-Meron Configuration

It is true, the Chern current $*j_c[\alpha]$ is not a linear construction of α but it incidentally satisfies the superposition law

$$*j_c[\alpha_a^+ + \alpha_b^+] = *j_c[\alpha_a^+] + *j_c[\alpha_b^+]. \quad \text{(III.16)}$$

Observe that the individual trivializable connections $\alpha_{a,b}$ must be substituted here in the positive gauge (II.11). For simplicity, we may imagine that each of them describes a single meron at a and b, resp., i.e.

$$\alpha_{a/b}^+ = \tfrac{1}{2}X_{a/b}^{-1}\cdot\mathbf{d}X_{a/b}, \quad \text{(III.17a)}$$

$$X_a = -\frac{x_\mu - a_\mu}{|x-a|}\,\varXi^\mu = -\hat{u}_\mu\,\varXi^\mu, \quad \text{(III.17b)}$$

$$X_b = -\frac{x_\mu - b_\mu}{|x-b|}\,\varXi^\mu = -\hat{v}_\mu\,\varXi^\mu. \quad \text{(III.17c)}$$

With these assumptions, the topological charge at the meron locations a or b becomes [15]

$$Q_{a/b} = \oint_{C_{a,b}^3} *j_c = +\tfrac{1}{2}, \quad \text{(III.18)}$$

where $C_{a,b}^3$ just encloses the corresponding meron position (but excludes the other one). The result (III.18) is immediately evident if one observes that the computation of the integrals has to be done in the positive gauge, where α^+ is half the value of a

trivial connection ($\alpha^+ = \frac{1}{2}\mathring{\alpha}$). This reduces the Chern current (III.5) to

$$*j_c[\tfrac{1}{2}\mathring{\alpha}] = (2\pi)^{-2}\,\mathbf{d}\,S^3[\hat{w}], \qquad (\text{III.19})$$

where the unit vector $\hat{w}$ has to be identified with $\hat{u}$ or $\hat{v}$, resp. Finally, we have the "orthogonality relation"

$$\oint_{C_a^3} *j_c[\alpha_b] = \tfrac{1}{2}\,\delta_{a,b}, \qquad (\text{III.20})$$

which immediately leads to the desired result (III.18). The conclusion from this result is that the connection $\alpha = \alpha_a^+ + \alpha_b^+$ describes two like merons at a and b, resp., where the total charge is given by $Q = Q_a + Q_b = 1$.

However, the gauge transformation mediated by the $SU^+(2)$ element X_b (III.17c) puts the original di-meron connection $\alpha = \alpha_a^+ + \alpha_b^+$ into the positive gauge

$$\alpha_{a,\bar{b}}^+ = \tfrac{1}{2}\,X_{a,\bar{b}}^{-1}\cdot\mathbf{d}\,X_{a,\bar{b}}, \qquad (X_{a,\bar{b}} = X_a \cdot X_b^{-1}). \qquad (\text{III.21})$$

Computing the topological charges in this gauge yields

$$Q_a = -Q_b = +\tfrac{1}{2} \qquad (\text{III.22})$$

in contrast to the previous result (III.18). The meron at b has become an anti-meron and hence the total charge changes from 1 into 0, though the gauge transformation applied is smooth over all E_4^-. We conclude from this result that, from the intrinsic point of view, one cannot say what is the difference between a meron and an anti-meron [17]. We are going to demonstrate now that this pathology may be cured partially by resorting to the extrinsic point of view via the bundle embedding.

III.4 Gauß Current

Consider the following gauge invariant 3-form $*j_G$ ("Gauß current") over E_4^-:

$$*j_G = \frac{1}{6\pi^2}\,\text{tr}\,\{\beta \wedge \beta \wedge \beta\}. \qquad (\text{III.23})$$

The Gauß current is closed on E_4^-,

$$\mathbf{d}\,*j_G = 0, \qquad (\text{III.24})$$

because of the covariant constancy condition (II.9) for β. In this respect, the Gauß current resembles very much the Chern class $*\varrho_c$ (III.1), which is closed on behalf of the Bianchi identity (II.8).

The geometric meaning of the Gauß current becomes immediately evident by the observation that anyone of the three 1-forms $\mathbb{B}_i$ ($i = 1, 2, 3$) annihilates the characteristic vector p by its very definition. Hence, if the gauge independent tensor field $B_{\mu\nu}$,

$$\hat{B}_{\mu\nu} := B_{i\mu}\,B_\nu^i, \qquad (\text{III.25})$$

constitutes a local automorphism $[\hat{B}]$: $\hat{\Delta} \to \hat{\Delta}$ of the "characteristic distribution" $\hat{\Delta}$ (with normal p) then $*j_G$ must be proportional to the "characteristic 3-form" $*\mathbb{P}$ (Poincaré dual of the characteristic vector p):

$$*j_G \sim *\mathbb{P}. \qquad (\text{III.26})$$

Concretely, this means that the characteristic lines are the flux lines of the Gauß current $*j_G$. If a given trivializable α admits the characteristic gauge (cf. (II.13)) then the general relation (III.26) may be further specialized by introducing the extrinsic curvature fields $\mathbb{B}_i$ (II.13) into the Gauß current $*j_G$ (III.23) which yields

$$*j_G = c^{-3}\,\frac{(p \cdot p)}{2\pi^2}\,*\mathbb{P}. \qquad (\text{III.27})$$

In defining $*j_G$ by (III.23), there remains an ambiguity in sign, because the replacement of β by $-\beta$ is allowed by the trivializability conditions (II.7 + 9). In other words: one has to decide what is the positive and what is the negative gauge! But apart from this overall ambiguity in sign, the topological charge Q_G, enclosed by some C^3,

$$Q_G[C^3] = \tfrac{1}{2}\oint_{C^3} *j_G, \qquad (\text{III.28})$$

is fixed and agrees with the first proposition $Q[C^3]$ (III.14) if the latter one is computed in the positive gauge. This assertion is immediately evident because the connection α agrees with the extrinsic curvature β in this gauge and hence the comparison of the two classes $*j_G$ (III.23) and $*j_c$ (III.5) yields

$$*j_c^+ = \tfrac{1}{2}\,*j_G. \qquad (\text{III.29})$$

In this sense, the positive gauge is preferred from the extrinsic point of view.

Simultaneously, the topological charges, fixed in this way, acquire a concrete meaning: since the general form of the extrinsic curvature coefficient

$\mathbb{B}_i$ is determined essentially by the unit normal $\hat{n}$ to $\bar{\varDelta}$,

$$\mathbb{B}_i = \hat{n} \cdot \mathbf{d}\,\hat{e}_i\,, \tag{III.30}$$

the Gauß current (III.23) becomes

$$*j_{\mathrm{G}} = \frac{1}{2\pi^2}\,\mathbf{d}\,\mathrm{S}^3[\hat{n}]\,, \tag{III.31}$$

which says that the charge Q_{G} is just the degree of the Gauß map $[\hat{n}]: C^3 \to S^3$. If $[\hat{n}]$ is a smooth map, the "winding number" $2\,Q_{\mathrm{G}}$ must be an integer and hence the topological charge itself is half-integer.

By reconsidering the example above, it is rather instructive to see in detail how the topological pathology of the intrinsic object $*j_{\mathrm{c}}$ is eliminated by the extrinsic object $*j_{\mathrm{G}}$. The characteristic vector $p(x)$ for the di-meron connection α (III.21) has been found as [1]

$$c^{-1}\,p(x) = \frac{x-a}{|x-a|^2} - \frac{x-b}{|x-b|^2} \tag{III.32}$$

and hence approaches the single meron shape if one comes close to one of the meron locations:

$$\begin{aligned} x \to a: \quad c^{-1}p &\to \frac{\hat{u}}{|x-a|}\,, \\ x \to b: \quad c^{-1}p &\to -\frac{\hat{v}}{|x-b|}\,. \end{aligned} \tag{III.33}$$

Therefore, the Gauß current $*j_{\mathrm{G}}$ (III.27) correctly says that there is a meron at a and an antimeron at b:

$$(x \to a): \quad *j_{\mathrm{G}} \to \frac{1}{2\pi^2}\,\mathbf{d}\,\mathrm{S}^3[\hat{u}]\,, \tag{III.34a}$$

$$(x \to b): \quad *j_{\mathrm{G}} \to -\frac{1}{2\pi^2}\,\mathbf{d}\,\mathrm{S}^3[\hat{v}] \tag{III.34b}$$

with the corresponding topological charges

$$Q_a = \frac{1}{(2\pi)^2}\,\oint_{C_a^3}\,\mathbf{d}\,\mathrm{S}^3[\hat{u}] = +\tfrac{1}{2}\,, \tag{III.35a}$$

$$Q_b = -\frac{1}{(2\pi)^2}\,\oint_{C_b^3}\,\mathbf{d}\,\mathrm{S}^3[\hat{v}] = -\tfrac{1}{2}\,. \tag{III.35b}$$

This is in contrast to the result (III.18), although both results have been obtained in the same gauge (cf. characteristic gauge). However, (III.35) agrees with (III.22) despite the fact that different gauges

have been used here (cf. positive versus characteristic gauge).

Clearly, one will prefer the extrinsic object $*j_{\mathrm{G}}$ over the ambiguous intrinsic object $*j_{\mathrm{c}}$ in order to be able to discern between like and unlike charges contained in a given multi-meron configuration. Let us remark that the problem of finding the correct current for the corresponding topological density does not arise for a SO(2) point charge in E_3^-, because the characteristic class of the SO(2) bundle is the Euler class, which is an element of $\mathrm{H}^2(\mathrm{E}_3^-, \mathbb{Z})$ and hence can be directly used to characterize the point defects [18]. In the present context, we shall use the Euler class in connection with the concept of a string in order to further elucidate the properties of the Gauß current.

III.5 Meron Pairing

For the present section, we assume the point defects to be of the meronic type i.e. the connection α may be put, by an appropriate gauge transformation, into the single meron shape (III.17a) in the neighborhood of the selected point defect. The characteristic lines emanating from such a meronic point defect have been identified with the flux lines of the Gauß current j_{G}, which measures the total topological charge Q, present in a N defect situation, via the charge integral (III.28). Clearly, the amount of the total charge Q might be smaller than the sum of the absolute values of the individual charges, because several of the flux lines emanating from some point defect may be absorbed by another point defect and thus are prevented from running out to infinity and contributing there to the total charge Q.

Hence, the question arises whether it is possible that some of the merons are "pairing", i.e. whether there exist couples of point defects such that all flux lines emanating from one member of the couple are absorbed completely by the other member [19]?

For the investigation of these pairing effects we consider a trivializable configuration, the generating matrix of which is of the form

$$X_{a,\bar{b},c,\dots} = X_a \cdot X_{\bar{b}}^{-1} \cdot X_c \cdot \dots = -\,\hat{n}_\mu\,\varXi^\mu\,, \tag{III.36a}$$

$$\hat{n}^\mu = \tfrac{1}{4}\,\mathrm{tr}\,\{\overset{+}{\varXi}{}^\mu \cdot X_{a,\bar{b},c,\dots}\}\,. \tag{III.36b}$$

Here, the individual SU$^+$(2) elements $X_a, X_{\bar{b}} = X_b^{-1}, X_c, \dots$ are just of the single meron shape

(III.17a) and hence the generating element $X_{a, \bar{b}, c, \ldots}$ (III.36) describes an ensemble of N individual merons located at $a, b, c, \ldots$. Apart form the special case $N = 2$, the gauge field corresponding to such an N defect configuration will not exactly satisfy the free Yang-Mills equations. But in the immediate neighborhood of any defect location M_a ($a = 1, \ldots, N$) the field equations are satisfied approximately and, hopefully, the exakt solution will exhibit merely some minor numerical deviations from the above product ansatz (III.36) off the defect locations. Especially, one may expect that the topological effect of pairing is truely described already by the approximate solution.

A first hint that the pairing effect really occurs is obtained by considering three single defects arranged along a straight line (x^0-axis, say). If the two like charges are neighbors on the symmetry axis, the two unlike charges are pairing and their common flux lines form a sort of bubble in the background field of the remaining single (Figure 1a). All the bubble lines are closed and contain the two members of the pair. The total charge Q ($= \frac{1}{2}$) is due to the remaining single defect whose flux lines run to infinity and thus are solely contributing to the charge integral.

It must be stressed however, that the pairing occurs only if the three defects form such an SO (3) symmetric alignment. If the single defect is pulled somewhat off the pair connecting line there arise flux lines which are connecting all three defects (Figure 1b).

The next question is whether there occurs pairing for an even number N of defects aligned alternatingly along the x^0-axis. Restricting ourselves to $N = 4$, we find that there is no pairing because besides the closed flux lines containing just one of the two pairs, there are always flux lines running through all four defects. This result holds for an arbitrary choice of distances between the point defects. For this reason, it does not seem meaningful to speak of meron pairing in this case (Fig. 2a) [21].

These results seem to hold also for an arbitrary N defect configuration. We have studied the case $N = 4$, when all defects are contained in the same 2-plane. The pairing effect only occurs here in a few highly symmetric configurations (Fig. 2b); in the general case, where the point defects are dislocated arbitrarily over the 2-plane, we never did observe the pairing phenomenon.
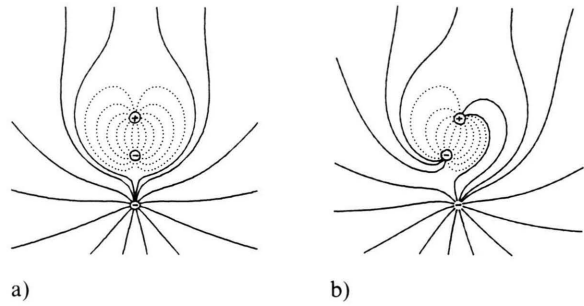

Fig. 1. Pairing of three point defects. The generating matrix X (III.36) produces a trivializable connection α, the positive gauge (α^+) of which is expressed by X as shown in (III.17a). The flux lines of the pair (dotted) are closed and therefore do not contribute to the total charge Q which is determined by the topological flux at infinity (Figure 1a). The latter one is due solely to the single defect (solid lines). The flux lines of the Gauß current are identical to the characteristic lines (tangent: p) along which the normal $\hat{n}$, obtained from X through (III.36b), is constant: $p_\mu \partial_\mu \hat{n}^\nu = 0$. However the pairing vanishes if the three point defects are not collinear, i.e. there arise flux lines which contain all three point defects (Figure 1b).
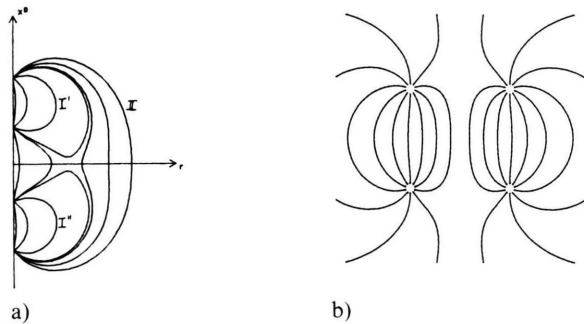

Fig. 2. Pairing of four point defects. If the four charges are aligned alternatingly along the symmetry axis, there is no pairing because there are always flux lines connecting all the point defects. For various choices of the Higgs vector parametrisation, there are two disjunct strings $\{I', I''\}$, or only a single string (II) (Figure 2a). The pairing effect also occurs for four merons (cf. two pairs) if they occupy the symmetric configuration shown in Figure 2b.

IV. Topological String Defects

Besides the meron configurations, there are other physically interesting applications of trivializable gauge fields, e.g. the SO (3) monopole configurations (t'Hooft-Polyakov monopole) which we shall study below. In these models, the original gauge group SO (3) is broken down to SO (2) by the effect of spontaneous symmetry breaking, and the macro-

scopic monopole field emerges as the Euler class of the reduced SO(2) bundle. It is essential for this mechanism of bundle reduction, that there exists a section $v(x)$ in the embedding SO(3) bundle $\bar{\tau}_4$ (the associated vector bundle of $\bar{\Sigma}_4$), by means of which the SO(2) reduction may be performed.

In the original t'Hooft-Polyakov model [22, 23], the required section v is incorporated into the Lagrangean through the Higgs field, which then plays the role of an extra dynamical variable besides the gauge field. However, we shall show hereafter that such a section arises very naturally in the context of a trivializable bundle connection. In turn, this "Higgs vector" v may then be used to parametrize the gauge field. The singularities of $v(x)$ determine some string $\mathcal{S}$, through which the topology of the embedding SO(3) bundle is linked to that of the reduced SO(2) bundle.

IV.1 Vector Parametrization

Remember the fact [1] that for any trivializable α, there always exists some $\bar{\tau}_4$ section $v(x)$ ("generating section"). In general, this section will not be uniquely determined but it can be used to parametrize the connection coefficients $\mathbb{A}_i$ in the neutral gauge as follows:

$$^\mathrm{n}\mathbb{A}_i = \left(1 - \cos\frac{v}{2}\right) \varepsilon_i{}^{jk}\, \bar{v}_k\, \mathbf{d}\, \bar{v}_j, \qquad (\mathrm{IV}.1)$$

$$(v^i = v\, \bar{v}^i,\ \bar{v}^i\, \bar{v}_i = -1).$$

Further, the extrinsic curvature coefficients $\mathbb{B}_i$ contain the covariant derivatives of the unit section $\bar{v}$ through

$$\mathbb{B}_i = \frac{1}{2}\, \bar{v}_i\, \mathbf{d}\, v + \tan\frac{v}{2}\, \mathbb{D}\, \bar{v}_i. \qquad (\mathrm{IV}.2)$$

Since all three 1-forms $\mathbb{B}_i$ $(i = 1, 2, 3)$ annihilate the characteristic vector p,

$$B_{i\mu}\, p^\mu = 0, \qquad (\mathrm{IV}.3)$$

the generating section $v(x)$ must be covariantly constant along any characteristic line:

$$p_\mu\, D_\mu\, v_i = 0. \qquad (\mathrm{IV}.4)$$

Moreover, in the neutral gauge it is also constant in the ordinary sense

$$p_\mu\, \partial_\mu\, v_i = 0. \qquad (\mathrm{IV}.5)$$

In topologically non-trivial situations the unit section $\bar{v}(x)$ and the scalar field $v(x)$ will develop some singular behaviour on a certain subset $\mathcal{S}$ of E_4. However, in view of the properties (IV.3–5) the singular set $\mathcal{S}$ must coincide with one (or some) of the characteristic lines. The nature of the singular behavior is assumed for the moment as

$$v_i\big|_{\mathcal{S}} = 0, \qquad (\mathrm{IV}.6)$$

i.e. $\bar{v}$ is undefined on $\mathcal{S}$ and the general 3-surface $v(x) = \mathrm{const}$ degenerates into the 1-dimensional submanifold $\mathcal{S} \subset E_4$ for $v(x) = 0$. Thus $\mathcal{S}$ will consist of a discrete set of disjoint characteristic lines, the number of which depends on the choice of the generating section $v(x)$. Remember that the choice of v results from the splitting of the 4-vector $\hat{n}^\mu(x)$, entering the generating $\mathrm{SU}^+(2)$ element X, into its "time and space components" according to

$$\hat{n}^\mu \rightarrow \left\{\cos\frac{v}{2};\ \bar{v}^i \sin\frac{v}{2}\right\}. \qquad (\mathrm{IV}.7)$$

Therefore, the freedom for choosing v may be traced back to the non-uniqueness of the distribution $\bar{\Delta}$ representing the trivializable α. Let us therefore have a short look at the uniqueness problem for $\bar{\Delta}$.

The right multiplication of X by a constant quaternion C,

$$X \rightarrow X' = X \cdot C = -\hat{n}'_\mu\, \Xi^\mu, \qquad (\mathrm{IV}.8\,\mathrm{a})$$

$$C = -\hat{c}_\mu\, \overset{+}{\Xi}{}^\mu, \qquad (\mathrm{IV}.8\,\mathrm{b})$$

gives a new matrix X', which however generates the same fibre bundle because the connection α thereby undergoes a constant gauge transformation (when considered in the positive gauge):

$$\alpha^{+\prime} = \tfrac{1}{2}\, (X \cdot C)^{-1} \cdot \mathbf{d}\, (X \cdot C) = C^{-1} \cdot \alpha^+ \cdot C. \qquad (\mathrm{IV}.9)$$

The normal $\hat{n}$ to the distribution $\bar{\Delta}$ experiences a constant SO(4) rotation according to

$$\hat{n}'_\sigma = \tfrac{1}{4}\, \mathrm{tr}\, \{\overset{+}{\Xi}_\sigma \cdot X'\} = \bar{C}^\mu{}_\sigma\, \hat{n}_\mu, \qquad (\mathrm{IV}.10)$$

where $\bar{C} \in \mathrm{SU}^-(2)$ is the anti-quaternion [1] of C:

$$\bar{C} = -\hat{c}_\mu\, \overset{+}{H}{}^\mu. \qquad (\mathrm{IV}.11)$$

Because the gauge transformation (IV.9) is described by a unit quaternion, we have

$$C \cdot C^{\mathrm{T}} = 1 \,, \tag{IV.12}$$

which implies that the unit normal $\hat{\boldsymbol{n}}$ is rotated into the vertical direction of group space on that characteristic line $\mathscr{S}'$ for which the original generating matrix X satisfies

$$C \cdot X\big|_{\mathscr{S}'} = 1 \,. \tag{IV.13}$$

Therefore, performing the decomposition (IV.7) for the rotated configuration (IV.10), one recognizes that the singularity condition (IV.6) shifts the original singular manifold $\mathscr{S}$ for $\boldsymbol{v}$ into the new set $\mathscr{S}'$ for $\boldsymbol{v}'$. As a consequence, the new gauge potential $^{n}\!A_i'$ arises from the old one (IV.1) by the substitution $\boldsymbol{v} \to \boldsymbol{v}'$ and becomes singular upon $\mathscr{S}'$ rather than on $\mathscr{S}$.

Obviously, the set of all C transformations (IV.8) is isomorphic to the gauge group SU(2) and therefore it is possible to select any of the characteristic lines, emanating from a point defect, as (at least part of) the singular manifold $\mathscr{S}$ [25]. On the other hand, any characteristic line starting at a point defect must terminate on an other point defect or has to run out to infinity; hence the singular set $\mathscr{S}$ will consist of a single characteristic line if all point defects $\mathscr{M}$ are contained in this line or otherwise $\mathscr{S}$ consists of several disjoint characteristic lines when the latter one form loops containing a few of the defect points $M_a \subset \mathscr{M}$. For the following considerations, we think of one (or several) of the characteristic lines be selected as the singular set $\mathscr{S}$, which is henceforth called the string [24].

As we shall see later, the string may be interpreted as monopole "world-line". Obviously, the introduction of the string is indispensible when we want to reduce the trivializable SO(3) bundle in order to get the reduced SO(2) bundle, in which the abelian monopole field lives as the bundle curvature. But from the point of view of the embedding SO(3) bundle, the string emerges as a spurious artefact, because the SO(3) objects are smooth over $E_4^- = E_4 \backslash \mathscr{M}$ and hence are smooth also on the string sections $\mathscr{S} \backslash \mathscr{M}$. For instance, we may look at the $\mathbb{B}$ fields (IV.2), the smoothness of which on $\mathscr{S} \backslash \mathscr{M}$ requires (generalizing (IV.6))

$$v\big|_{\mathscr{S} \backslash \mathscr{M}} = n \cdot 2\pi \,, \quad n \in \mathbb{Z} \,. \tag{IV.14}$$

The integer n is expected to change discontinuously along the string $\mathscr{S}$ when one passes anyone of the point defect locations $M_a \in \mathscr{S}$. Clearly, one supposes that this change of the integer n is related to the topological charge Q (III.14, 28) residing at M_a. We are going to clarify this relationship now by composing the topological invariant Q of the Euler number of the reduced SO(2) bundle and the change of n.

IV.2 Euler Class

Once a section $\boldsymbol{v}$ is given in the vector bundle $\bar{\tau}_4$, one can consider the bundle reduction along this section. The curvature in the reduced SO(2) bundle is the well-known Euler class [26], which reads in terms of the unit section $\bar{v} := \bar{v}_i X^i$ and $\mathfrak{SU}^+(2)$ curvature Φ (II.7a)

$$\tilde{\mathbb{F}} = \mathrm{tr}\{\bar{v}\,\Phi - \bar{v}(D\,\bar{v}) \wedge (D\,\bar{v})\} \,. \tag{IV.15}$$

One is easily convinced that $\tilde{\mathbb{F}}$ just describes the field of an abelian monopole with quantized charge $\tilde{g}$. Indeed introducing the curvature Φ (II.7a) of a (general) gauge field α into (IV.15) for $\tilde{\mathbb{F}}$ immediately yields

$$\tilde{\mathbb{F}} = \mathbf{d}\tilde{\mathbb{A}} - \mathbf{d}S^2 \,, \tag{IV.16a}$$

$$\tilde{\mathbb{A}} = \mathrm{tr}\{\bar{v} \cdot \alpha\} \,, \tag{IV.16b}$$

$$\mathbf{d}S^2 = \mathrm{tr}\{\bar{v} \cdot (\mathbf{d}\bar{v}) \wedge (\mathbf{d}\bar{v})\} \,. \tag{IV.16c}$$

Here, $\mathbf{d}S^2[\bar{v}]$ is the pullback of the 2-cell on the unit sphere S^2 with respect to the map $[\bar{v}]: E_4 \backslash \mathscr{S} \to S^2$, mediated by the generating section $\boldsymbol{v}$.

The quantization condition upon the abelian charge $\tilde{g}\,[C^2]$ is found readily by taking the value of $\tilde{\mathbb{F}}$ upon some 2-cycle enclosing the "monopole world line" $\mathscr{S}$:

$$\tilde{g}\,[C^2] = \frac{1}{4\pi} \oint_{C^2} \tilde{\mathbb{F}} = -\tilde{n}\,, \quad \tilde{n} \in \mathbb{Z} \,. \tag{IV.17}$$

Since the first term on the right of (IV.16a) is a total derivative, it does not contribute to the charge integral (IV.17). Moreover, this term is absent for a trivializable α if the neutral gauge (IV.1) is used.

Clearly, the Euler class is an element of the second cohomology group $H^2(E_4 \backslash \mathscr{S}, \mathbb{Z})$ and hence is closed on $E_4 \backslash \mathscr{S}$:

$$\mathbf{d}\tilde{\mathbb{F}} = 0 \,. \tag{IV.18}$$

But this immediately implies that the charge $\tilde{g}$ (IV.17) does not change when C^2 is moving along

the string $\mathscr{S}$. Hence, the monopole charge $\tilde{g}$ is a topological invariant of the SO(2) gauge field configuration. Now the question arises how the SO(2) charge $\tilde{g}$ is related to the SU(2) charge Q?

In order to answer this question briefly, we simply introduce the vector parametrization (IV.2) of the extrinsic curvature coefficients $\mathbb{B}_i$ into the Gauß current $*j_G$ (III.23) and express it in terms of the Euler class $\tilde{\mathbb{F}}$ and the length $v(x)$ of the generating section

$$*j_G = -\frac{1}{2\pi^2}\,\tilde{\mathbb{F}} \wedge \mathbf{d}\,I_{(v)}\,, \qquad (IV.19\,a)$$

$$I_{(v)} = \tfrac{1}{4}\,(v - \sin v)\,. \qquad (IV.19\,b)$$

According to the presence of a wedge product in $*j_G$ (IV.19a), the charge integral Q (III.28) factorizes according to the foliation of C^3 into the 2-cycles $C^2_{(v)}$ ($v|_{C^2_{(v)}} = \text{const}$) and the complementary lines connecting the intersections $C^3 \cap \mathscr{S}$:

$$Q = -\frac{1}{2\pi^2}\,\oint_{C^2_{(v)}}\tilde{\mathbb{F}}\,\int_{v_-}^{v_+}\mathbf{d}\,I_{(v)}\,. \qquad (IV.20)$$

If we attribute an orientation to the string such that $v|_{\mathscr{S}}$ increases along the positive direction ($v_+ > v_-$), this charge integral becomes a well-defined half-integer as expected and is expressed by the abelian charge $\tilde{g}$ and the integer n occurring in the smoothness condition (IV.14) as follows:

$$Q = \tfrac{1}{2}\,\tilde{g}\,(n_+ - n_-)\,. \qquad (IV.21)$$

Thus, the SU(2) charge Q is determined by both the abelian charge $\tilde{g}$ and the total change of the string variable $v|_{\mathscr{S}}$ across C^3. Because the non-trivial minimal values of both constituents are given by

$$|\tilde{g}| \geqq 1\,, \qquad |n_+ - n_-| \geqq 1 \qquad (IV.22)$$

the minimal value of the topological charge Q is just the meron value (1/2). Further, since the abelian charge $\tilde{g}$ does not change along the same string $\mathscr{S}$, the different charges Q_a along $\mathscr{S}$ must be solely due to the different changes of the values of the integer n in the neighborhood of a defect location $M_a \in \mathscr{S}$. For this reason, the topological charge Q may be attributed to a certain string flux j_s, which has its sinks and sources just in the defect locations M_a but vanishes off $\mathscr{S}$. In order to make this approach somewhat more precise, we consider

the continuation of the "Maxwell equation" (IV.18) over the whole E_4, where it appears as

$$\mathbf{d}\,\tilde{\mathbb{F}} = -4\pi\,*j_M\,. \qquad (IV.23)$$

Here, the Maxwell current $*j_M$ is due to a point like SO(2) monopole of charge $\tilde{g}$:

$$j_M(x) = \tilde{g}\,\int_{-\infty}^{\infty}\mathrm{d}s\,\hat{t}\,\delta(x - z(s))\,. \qquad (IV.24)$$

Its world line is just the string $\mathscr{S}$,

$$\mathscr{S}:\ x = z(s)\,, \qquad (IV.25)$$

which is parametrized by the "proper time" s corresponding to the arc length on $\mathscr{S}$ [26]. The unit tangent $\hat{t}\ (\equiv \mathrm{d}z/\mathrm{d}s)$ to $\mathscr{S}$ plays the part of "four-velocity" of the monopole [27]. Now, integrating the Gauß current (IV.19) by parts, the exterior derivative of the Euler class $\tilde{\mathbb{F}}$ enters the current expression and may be substituted from the Maxwell equation (IV.23). As a consequence, the Gauß current becomes the sum of an exact differential $\mathbf{f}$ and of the string flux $*j_s$, which is built up essentially by the Maxwell current $*j_M$:

$$*j_G = \mathbf{d}\,\mathbf{f} + *j_s\,, \qquad (IV.26\,a)$$

$$\mathbf{f} = -\frac{1}{2\pi^2}\,I_{(v)}\,\tilde{\mathbb{F}}\,, \qquad (IV.26\,b)$$

$$*j_s = I_{(v)}\,*j_M = \tilde{g}\,\int_{-\infty}^{\infty}\mathrm{d}s\,n(s)\,\hat{t}(s)\,\delta(x - z(s))\,. \qquad (IV.26\,c)$$

But the exact differential $\mathbf{d}\,\mathbf{f}$ does not contribute to the charge integral, which then is solely due to the string flux $*j_s$:

$$Q[C^3] = \tfrac{1}{2}\,\oint_{C^3}*j_s = \tfrac{1}{2}\,\tilde{g}\,\sum_{C^3 \cap \mathscr{S}}n\,. \qquad (IV.27)$$

The contributions to the sum of integers n come from the points of intersection of the string $\mathscr{S}$ and C^3. Hence, the topological charge Q enclosed by C^3 is proportional to the total change of n across C^3 (Figure 3).

V. Monopole Configurations

The "static" field configurations of the trivializable type are characterized by the requirement that the characteristic vector $\boldsymbol{p}$ be a parallel field with
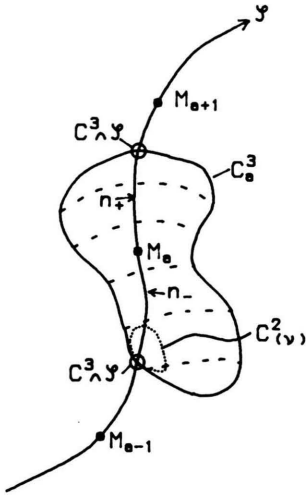
Fig. 3. Abelian and non-abelian topological defects. The monopole world-line $\mathscr{S}$ (IV.25) is a selected characteristic line and contains the defect location M_a at which the non-abelian charge Q resides. The latter one is the product of the abelian charge $\tilde{g}$ (IV.17) and the change $(n_+ - n_-)$ of the integer n across the defect location M_a: $Q_a = \frac{1}{2} \tilde{g} \, (n_+ - n_-)$ (cf. (IV.20)). The charge Q may be concretely attributed to an abelian string flux $*j_s$ (IV.26c), which has a sink or source at the defect location M_a. The monopole charge $\tilde{g}$ is topologically the Euler number of the reduced SO (2) bundle and is given by the value of the Euler class $\mathbb{F}$ (IV.15) upon any 2-cycle $C^2_{(v)}$ which is the intersection of a 3-chain $v(x) = $ const and the 3-cycle C^3_a enclosing M_a (cf. (IV.20)).

respect to the canonical connection $\mathring{\omega}$ over E_4, i.e. its derivative is of the shape

$$\partial_\mu p^\nu = s_\mu p^\nu \qquad (V.1)$$

with an arbitrary 1-form s_μ. Without loss of generality, we may choose the x^0-axis of some Cartesian coordination of E_4 as the direction of the translational invariance induced by the parallel p field. Consequently, the whole field geometry becomes "static" in the sense

$$B_{i0} = 0, \qquad (V.2a)$$

$$p = p \, \vec{\partial}_0. \qquad (V.2b)$$

Obviously, the physical implication of this assumption is that one is dealing with monopoles which are "at rest" relative to the chosen frame. Moreover, we shall restrict ourselves to SO (3) symmetric configurations such that the string $\mathscr{S}$ becomes identical to the x^0-axis of the Cartesian frame. Before studying the corresponding monopole geometry we first have a look at the asymptotic conditions.

V.1 Higgs Vacuum

Usually, one imposes the condition of an asymptotic Higgs vacuum when one tries to get a monopole solution to the coupled Yang-Mills-Higgs system. Therefore, it appears meaningful to consider first the geometric and topological properties of such a Higgs vacuum phase.

An immediate geometric approach to this phase is obtained by reconsidering the Gauß current j_G for the present situation:

$$j_G^\sigma = \frac{1}{(2\pi)^2} \left(\frac{\sin \frac{v}{2}}{r} \right)^2 \dot{v} \, \hat{t}^\sigma, \qquad (V.3)$$

$$(\dot{v} := \mathrm{d}v/\mathrm{d}r, \ \hat{t} \equiv \vec{\partial}_0).$$

If this expression is used to define the static analogue $\tilde{Q}$ of the topological charge Q (III.28) through

$$\tilde{Q} = \frac{1}{2} \int_{x^0 = \mathrm{const}} *j_G \qquad (V.4)$$

one readily recognizes the importance of the boundary values of the angular variable $v(r)$:

$$\tilde{Q} = \frac{1}{4\pi} \left. (v - \sin v) \right|_{r=0}^{r=\infty}. \qquad (V.5)$$

Thus, a topologically non-trivial monopole solution will arise, whenever the following requirement is fulfilled:

$$v_{(0)} = 0, \qquad (V.6a)$$

$$v_{(r \to \infty)} = 2\pi \tilde{n}, \quad \tilde{n} \in \mathbb{Z}. \qquad (V.6b)$$

For in this case the Euclidean 3-space E_3 ($x^0 = $ const) is mapped, via the normal $\hat{n}(v)$, into the sphere S^3. This essentially is equivalent to a compactification of E_3 whenever (V.6b) holds. In this case, the static charge $\tilde{Q}$ (V.4) acquires the meaning of the previous topological charge Q (III.28) and hence assumes half-integer values. The minimal "compactifying" value agrees then with the meron case ($\tilde{Q} \to 1/2$).

However, let us point out that there is also the possibility of a "half-meron" ($\tilde{Q} = 1/4$), where $\tilde{n} = 1/2$. Such an asymptotic configuration is called hereafter a "Higgs-vacuum". The notation comes from the fact that for $v = \pi$ the covariant derivative of the generating unit section $\bar{v}(x)$ vanishes (verify it in the neutral gauge),

$$D_\mu \bar{v}^i = 0, \qquad (V.7)$$

and hence the gauge potential $A_{i\mu}$ assumes the well-known shape due to a Higgs vacuum [29]

$$A_{i\mu} \to {}^{H}A_{i\mu} = \varepsilon_i{}^{jk} \bar{v}_k \, \partial_\mu \bar{v}_j. \qquad (V.8)$$

Further, the external curvature becomes proportional to the projector $\tilde{P}_{ij}$ $(= g_{ij} + \bar{v}_i \bar{v}_j)$ along the unit section

$$B_{ij} = \tilde{P}_{ij}/r^2. \qquad (V.9)$$

The curvature of the vacuum potential ${}^{H}\!\!\textbf{A}_i$ (V.8) explicitly contains the SO(2) monopole field $\tilde{\mathbb{F}}$ (Euler class) through

$${}^{H}\tilde{\mathbb{F}}_i = -\bar{v}_i \, \tilde{\mathbb{F}}. \qquad (V.10)$$

From the geometric point of view, the Higgs vacuum phase is represented by a system of 3-cylinders $\{Z^3\}$ the generating lines of which are parallel to the monopole world line $\mathscr{S}$. The intersections of the cylinders with the sheet $x^0 = $ const are 2-spheres centered around the monopol position. The desired solution to the Yang-Mills-Higgs system is now represented by a system of 3-surfaces $\{M^3\}$ which approach the cylinders $\{Z^3\}$ at "spacelike" infinity (Figure 4). Observe however that the solution surfaces $\{M^3\}$ are mapped onto one half of the unit sphere S^3 $(0 < v \leq \pi)$; thus the image of M^3 on S^3 can be contracted into a single point and hence the solution M^3 is not topologically stable in the strict sense [30]. It must be stabilized dynamically by some symmetry breaking potential in the Lagrangean, the minimum of which fixes the absolute length of the Higgs field in the asymptotic vacuum phase. The degeneracy of the potential minimum then transfers its non-trivial topology to the gauge field via (V.8).

V.2 t'Hooft-Polyakov Monopole

A famous example of a static trivializable gauge field of the topologically unstable type is the t'Hooft-Polyakov monopole solution to the coupled Yang-Mills-Higgs system

$$D^\mu F_{i\mu\nu} = \varepsilon_i{}^{jk} \Phi_k D_\nu \Phi_j,$$

$$D^\mu D_\mu \Phi_i = \partial V / \partial \Phi^i,$$

$$V(\Phi) = (\lambda/2)^2 \{\Phi_\infty^2 - \Phi^2\},$$

$$(\Phi^2 := -\Phi^i \Phi_i, \ \Phi_\infty = \text{const}). \qquad (V.11)$$

For an SO(3) symmetric solution of (V.11) one will try the 3-dimensional hedgehog ansatz for the Higgs field Φ_i:

$$\Phi_i = \Phi_{(r)} \bar{x}_i, \ (x_i = r \, \bar{x}_i, \, \bar{x}^i \bar{x}_i = -1), \qquad (V.12)$$

whereas the gauge field may be used in the neutral gauge (IV.1). The coupled system (V.11) then reduces to

$$2 \tan \frac{v}{2} \ddot{v} + \dot{v}^2 - 2 \left(\frac{\sin \frac{v}{2}}{r} \right)^2 = -(2\Phi)^2, \qquad (V.13a)$$

$$\ddot{\Phi} + \frac{2}{r} \dot{\Phi} - 2\Phi \left(\frac{\cos \frac{v}{2}}{r} \right)^2 = \lambda^2 \Phi (\Phi^2 - \Phi_\infty^2). \qquad (V.13b)$$

An exact solution is known for $\lambda = 0$ (Prasad-Sommerfield limit) [31], namely

$$\Phi/\Phi_\infty = \text{Coth}(r\,\Phi_\infty) - 1/r\,\Phi_\infty, \qquad (V.14a)$$

$$\cos \frac{v}{2} = \frac{r\,\Phi_\infty}{\text{Sinh}(r\,\Phi_\infty)}. \qquad (V.14b)$$

This solution satisfies the trivializable ansatz and hence its gauge field part must be representable by a static 3-distribution $\bar{A}$ over E_4 (see Fig. 4 for $\lambda = 0$). For other values of λ numerical solutions with
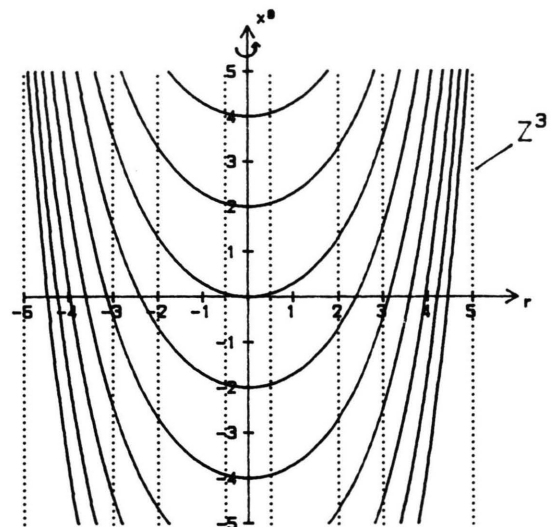

Fig. 4. The t'Hooft-Polyakov solution. The 3-surfaces M^3 carry the monopole potential $\textbf{A}_i$ as the Levi-Civita connection of the Riemannian 3-metric induced by the surface restriction of the Euclidean 4-metric g of the embedding E_4. The monopole surfaces M^3 asymptotically approach the 3-cylinders Z^3, which geometrically represent the Higgs vaccum.

a finite action may be obtained which also exhibit the property of trivializability.

In this way, we come to the result that the SO (3) symmetric finite action solutions to the Yang-Mills-Higgs system just agree with the set of its trivializable solutions. We have checked numerically also the converse statement and we did not find any

smooth infinite action solution to (V.13) for $0 < r < \infty$, which was trivializable at the same time. Unfortunately, we are unable to show whether this interesting relationship between the physical requirement (finite action) and the geometric property (trivializability) also holds in more general situations (non-symmetric, non-static solutions).

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